

SOLUTIONS AND STABILITY OF A GENERALIZATION OF WILSON'S EQUATION

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ABSTRACT. In this paper we study the solutions and stability of the generalized Wilson's functional equation $\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y)$, $x, y \in G$, where G is a locally compact group, σ is a continuous involution of G and μ is an idempotent complex measure with compact support and which is σ -invariant. We show that $\int_G g(xty)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) = 2g(x)g(y)$, $x, y \in G$ if $f \neq 0$ and $\int_G f(t.)d\mu(t) \neq 0$. We also study some stability theorems of that equation and we establish the stability on noncommutative groups of the classical Wilson's functional equation $f(xy) + \chi(y)f(x\sigma(y)) = 2f(x)g(y)$, $x, y \in G$, where χ is a unitary character of G .

1. INTRODUCTION AND PRELIMINARIES

Although d'Alembert's functional equation

$$(1.1) \quad f(x+y) + f(x-y) = 2f(x)f(y) \text{ for all } x, y \in \mathbb{R}$$

for functions $f: \mathbb{R} \rightarrow \mathbb{C}$ on the real line has its roots back in d'Alembert's investigation of vibrating strings [10] from 1775. Furthermore, one solution of (1.1) is $f(x) = \cos(x)$, another $f(x) = \cosh(x)$. The obvious extension of (1.1) from \mathbb{R} to an abelian group $(G, +)$ is the functional equation

$$(1.2) \quad f(x+y) + f(x-y) = 2f(x)f(y) \text{ for all } x, y \in G,$$

where $f: G \rightarrow \mathbb{C}$ is the unknown. The non-zero solution of equation (1.2) are of the form $f(x) = \frac{\chi(x) + \chi(-x)}{2}$, $x \in G$, where χ is a character in G . The result is obtained by Kannappan in [18]. If the group G is not assumed abelian the solutions of equation

$$(1.3) \quad f(xy) + f(xy^{-1}) = 2f(x)f(y) \text{ for all } x, y \in G$$

are obtained by Davison [11, 12]. There are of the form $f = \frac{1}{2}tr(\varrho)$, where ϱ is continuous algebraically irreducible representation of G on \mathbb{C}^2 .

In [31] Wilson dealt with functional equations related to and generalizing

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(1.1) on the real line. He generalized the d'Alembert's functional equation (1.1) to

$$(1.4) \quad f(x+y) + f(x-y) = 2f(x)g(y) \text{ for all } x, y \in \mathbb{R}.$$

Let us note that if $f \neq 0$ is a solution of equation (1.4), then g satisfies equation (1.1).

Some general properties of the solutions of equation

$$(1.5) \quad f(xy) + f(x\sigma(y)) = 2f(x)g(y) \text{ for all } x, y \in G$$

on a topological monoid equipped with a continuous involution σ can be found in [26].

Recently, Ebanks, Stetkær [13] and Stetkær [28] proved a natural interesting relation between Wilson's functional equation (1.5) and d'Alembert's functional equation

$$(1.6) \quad f(xy) + f(x\sigma(y)) = 2f(x)f(y) \text{ for all } x, y \in G$$

and for $\sigma(x) = x^{-1}$. That is if $f \neq 0$ is a solution of equation (1.5), the g is a solution of equation (1.6).

The Hyers-Ulam stability of d'Alembert's functional equation (1.2) was investigated by J.A. Baker in [4]. In [5], J. Baker, J. Lawrence and F. Zorzitto introduced the superstability of the exponential equation $f(x+y) = f(x)f(y)$, $x, y \in G$. Badora [1] gave a new, shorter proof of Baker's result. A different generalization of the result of Baker, Lawrence and Zorzitto was given by L. Székelyhidi [30].

On abelian groups, the stability of d'Alembert's functional equation (1.2) and Wilson's functional equation (1.4) and other functional equation has been investigated by several authors. The interested reader should refer to [2], [3], [6], [9], [15], [16], [17], [20], [21], [22], [23], [24], [25], [29] and [32], for a thorough account on the subject of stability of functional equations.

The aim of this paper is to study some properties of the solutions and Hyers-Ulam stability of some generalization of d'Alembert's and Wilson's functional equations which has been introduced in [14]. As an application we obtain the Hyers-Ulam stability of Wilson's functional equation (1.5) on groups that need not be abelian.

Throughout this paper, we let G be a locally compact group, $C(G)$ the complex algebra of all continuous complex valued functions on G . $M(G)$ the Banach algebra of the complex bounded measures on G . It's the topological dual of $C_0(G)$: The Banach space of continuous functions vanishing at infinity. Let $\sigma: G \rightarrow G$ be a continuous involution of G , that is $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$. If $\mu \in M(G)$ is a measure with compact support, we let μ_σ denote the complex measure with compact support and defined by the relation: $\langle \mu_\sigma, f \rangle = \langle \mu, f \circ \sigma \rangle$, $f \in C(G)$, where $\langle \mu, f \rangle = \int_G f(t) d\mu(t)$. We will say that μ is σ -invariant if $\mu = \mu_\sigma$. We recall that the convolution measure $\mu * \mu$ is the measure defined on $C(G)$ by $\langle \mu * \mu, f \rangle = \int_G \int_G f(ts) d\mu(t) d\mu(s)$. Finally, for a

continuous function f we let $f_\mu(x) = \int_G f(tx)d\mu(t)$, $x \in G$ and we say that f is μ -biinvariant, if $\int_G f(xt)d\mu(t) = \int_G f(tx)d\mu(t) = f(x)$, for all $x \in G$.

2. RELATIONS BETWEEN WILSON'S AND D'ALEMBERT'S FUNCTIONAL EQUATIONS

In the special case where μ is a Gelfand measure or f satisfies some Kannappan type condition Elqorachi and Akkouchi [[14], Proposition 3.2] obtained a natural relation between the generalized Wilson's functional equation

$$(2.1) \quad \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y), \quad x, y \in G$$

and the generalized d'Alembert's short functional equation

$$(2.2) \quad \int_G g(xty)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) = 2g(x)g(y), \quad x, y \in G$$

That is if the pair $f, g: G \rightarrow \mathbb{C}$, where $f \neq 0$, is a solution of generalized Wilson's functional equation (2.1) then g is a solution of the generalized d'Alembert's functional equation (2.2). In more general setting the authors [[8], Corollary 2.7 (iii)] got a weaker result, that is if the pair $f, g: G \rightarrow \mathbb{C}$, where $f \neq 0$, is a solution of generalized Wilson's functional equation (2.1) then g is a solution of the generalized d'Alembert's long functional equation (2.3)

$$\begin{aligned} \int_G g(xty)d\mu(t) + \int_G g(ytx)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) \\ = 4g(x)g(y), \quad x, y \in G. \end{aligned}$$

The following theorem is a generalization of the result obtained by Ebanks, Stetkær [13].

Theorem 2.1. *Let σ be a continuous involution of G . Let μ be a complex measure with compact support and which is σ -invariant. If the pair $f, g: G \rightarrow \mathbb{C}$, where $f \neq 0$ is a continuous solution of the generalized Wilson's functional equation (2.1) and f is odd: $f(\sigma(x)) = -f(x)$ for all $x \in G$. Then g is a solution of the generalized d'Alembert's short functional equation (2.2).*

Proof. The proof is closely related to the one obtained by Ebanks, Stetkær [13]. Let us assume that the pair f, g is a solution of equation (2.1) with $f \neq 0$ and $f(\sigma(x)) = -f(x)$ for all $x \in G$. By replacing y by $\sigma(y)$ in (2.1) we get $g(\sigma(x)) = g(x)$ for all $x \in G$. Now, we consider the new function

$$\Psi(x, y) = \int_G g(xty)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) - 2g(x)g(y), \quad x, y \in G.$$

By using (2.1) we obtain

$$2f(z)\Psi(x, y) + 2f(y)\Psi(x, z) = 2f(z) \left[\int_G g(xty)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) - 2g(x)g(y) \right]$$

$$\begin{aligned}
& +2f(y)[\int_G g(xtz)d\mu(t) + \int_G g(\sigma(z)tx)d\mu(t) - 2g(x)g(z)] \\
& = \int_G \int_G f(zsxt y)d\mu(t)d\mu(s) + \int_G \int_G f(zs\sigma(y)\sigma(t)\sigma(x))d\mu(t)d\mu(s) \\
& + \int_G \int_G f(zs\sigma(y)tx)d\mu(t)d\mu(s) + \int_G \int_G f(zs\sigma(x)\sigma(t)y)d\mu(t)d\mu(s) \\
& - \int_G \int_G f(ztxsy)d\mu(t)d\mu(s) - \int_G \int_G f(zsxt\sigma(y))d\mu(t)d\mu(s) \\
& - \int_G \int_G f(zt\sigma(x)sy)d\mu(t)d\mu(s) - \int_G \int_G f(zt\sigma(x)s\sigma(y))d\mu(t)d\mu(s) \\
& + \int_G \int_G f(ysxtz)d\mu(t)d\mu(s) + \int_G \int_G f(ys\sigma(z)\sigma(t)\sigma(x))d\mu(t)d\mu(s) \\
& + \int_G \int_G f(ys\sigma(z)tx)d\mu(t)d\mu(s) + \int_G \int_G f(ys\sigma(x)\sigma(t)z)d\mu(t)d\mu(s) \\
& - \int_G \int_G f(ytxsz)d\mu(t)d\mu(s) - \int_G \int_G f(ytxs\sigma(z))d\mu(t)d\mu(s) \\
& - \int_G \int_G f(yt\sigma(x)sz)d\mu(t)d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))d\mu(t)d\mu(s) \\
& = \int_G \int_G f(zt\sigma(y)s\sigma(x))d\mu(t)d\mu(s) + \int_G \int_G f(zt\sigma(y)sx)d\mu(t)d\mu(s) \\
& + \int_G \int_G f(yt\sigma(z)s\sigma(x))d\mu(t)d\mu(s) + \int_G \int_G f(yt\sigma(z)sx)d\mu(t)d\mu(s) \\
& - \int_G \int_G f(zsxt\sigma(y))d\mu(t)d\mu(s) - \int_G \int_G f(zs\sigma(x)t\sigma(y))d\mu(t)d\mu(s) \\
& - \int_G \int_G f(ytxs\sigma(z))d\mu(t)d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))d\mu(t)d\mu(s),
\end{aligned}$$

where the last identity is due to our assumption that μ is σ -invariant. Using equation (2.1) and the above computation to obtain

$$\begin{aligned}
& 2f(z)\Psi(x, y) + 2f(y)\Psi(x, z) \\
& = 2g(x) \int_G f(zt\sigma(y))d\mu(t) + 2g(x) \int_G f(yt\sigma(z))d\mu(t) \\
& - \int_G \int_G f(ytxs\sigma(z))d\mu(t)d\mu(s) - \int_G \int_G f(zs\sigma(x)t\sigma(y))d\mu(t)d\mu(s) \\
& - \int_G \int_G f(zsxt\sigma(y))d\mu(t)d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))d\mu(t)d\mu(s) \\
& = 2g(x) \int_G f(zt\sigma(y))d\mu(t) - 2g(x) \int_G f(zt\sigma(y))d\mu(t) \\
& + \int_G \int_G f(zs\sigma(x)t\sigma(y))d\mu(t)d\mu(s) - \int_G \int_G f(zs\sigma(x)t\sigma(y))d\mu(t)d\mu(s) \\
& + \int_G \int_G f(yt\sigma(x)s\sigma(z))d\mu(t)d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))d\mu(t)d\mu(s) = 0,
\end{aligned}$$

which is due to assumptions that f is odd and μ is σ -invariant. So, this implies that $f(z)\Psi(x, y) + f(y)\Psi(x, z) = 0$ for all $x, y, z \in G$ and then we conclude that there exists $c_x \in \mathbb{C}$ such that $\Psi(x, y) = c_x f(y)$, $c_x f(z)f(y) + c_x f(y)f(z) = 0$. We get for any $x, y \in G$ $\Psi(x, y) = 0$. Now, since g is even: $g(\sigma(x)) = g(x)$ for all $x \in G$ and μ is σ -invariant we obtain

$$\begin{aligned} 2g(x)g(y) &= \int_G g(xty)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) \\ &= \int_G g(\sigma(y)tx)d\mu(t) + \int_G g(\sigma(y)t\sigma(x))d\mu(t) = 2g(\sigma(y))g(x). \end{aligned}$$

This means that

$$\int_G g(xty)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) = 2g(x)g(y)$$

for all $x, y \in G$ and this completes the proof.

Theorem 2.2. *Let σ be a continuous involution of G . Let μ be a complex measure with compact support such $\mu * \mu = \mu$ and μ is σ -invariant. If the pair $f, g: G \rightarrow \mathbb{C}$, where $f \neq 0$ is a continuous solution of the generalized Wilson's functional equation (2.1) such that $f_\mu \neq 0$. Then g is a solution of the generalized d'Alembert's short functional equation (2.2).*

In the proof we use the ideas of Stetkær [28]. Let g be a non zero fixed solution of equation (2.1).

We put $W_g := \{f : G \rightarrow \mathbb{C} \mid f \text{ is continuous, satisfies (2.1), } f_\mu = f \text{ and } f(e) = 0\}$. A continuous solution f of equation (2.1) such that $f_\mu = f$ is odd iff $f(e) = 0$.

If $W_g \neq \{0\}$ then we get the result from Theorem 2.1. Assume now that $W_g = \{0\}$. Let f be a non zero solution of equation (2.1) such that $f_\mu \neq 0$. The function f_μ is also a nonzero solution of equation (2.1). Since $W_g = \{0\}$, $(f_\mu)_\mu = f_\mu$, then $f_\mu(e) \neq 0$. Replacing f_μ by $f_\mu/f_\mu(e)$ we may assume that $f_\mu(e) = 1$. If h is a continuous solution of equation (2.1), then $h_\mu - h_\mu(e)f_\mu \in W_g = \{0\}$, so $h_\mu = h_\mu(e)f_\mu$. Let $x \in G$, since $\delta(x)f_\mu(y) = \int_G f_\mu(xty)d\mu(t)$ is a solution of equation (2.1), $\mu * \mu = \mu$ and $(\delta(x)f_\mu)_\mu = \delta(x)f_\mu$, then there exists $\psi(x)$ such that $\delta(x)f_\mu = \psi(x)f_\mu$. In particular for $y = e$ we have $(\psi(x)f_\mu)(e) = \psi(x) = (\delta(x)f_\mu)(e) = \int_G f_\mu(xt)d\mu(t)$, so we get

$$(2.4) \quad \int_G f_\mu(xty)d\mu(t) = f_\mu(y) \int_G f_\mu(xt)d\mu(t)$$

for all $x, y \in G$.

Now, we will show that $\int_G f_\mu(xt)d\mu(t) = f_\mu(x)$ for all $x \in G$. Since f_μ satisfies (2.1) and $\mu * \mu = \mu$ then we get

$$\begin{aligned} &\int_G f_\mu(xty)d\mu(t) + \int_G f_\mu(xt\sigma(y))d\mu(t) = 2f_\mu(x)g(y) \\ &= \int_G \int_G f_\mu(xsty)d\mu(s)d\mu(t) + \int_G \int_G f_\mu(xst\sigma(y))d\mu(s)d\mu(t) = 2 \int_G f_\mu(xs)d\mu(s)g(y), \end{aligned}$$

which implies that $\int_G f_\mu(xs)d\mu(s) = f_\mu(x)$, for all $x \in G$ and then equation (2.4) can be written as follows. $\int_G f_\mu(xty)d\mu(t) = f_\mu(x)f_\mu(y)$ for all $x, y \in G$. Substituting this result into

$$\int_G f_\mu(xty)d\mu(t) + \int_G f_\mu(xt\sigma(y))d\mu(t) = 2f_\mu(x)g(y)$$

we get $g(y) = \frac{f_\mu(y) + f_\mu(\sigma(y))}{2}$ and we can easily verify that g satisfies the generalized d'Alembert's short functional equation (2.2). This ends the proof of theorem. \square

3. HYERS-ULAM STABILITY OF WILSON'S FUNCTIONAL EQUATION

In [[8], Corollary 2.7 (iii)] the authors proved that if the function

$$(x, y) \longrightarrow \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)$$

is bounded and f is unbounded then g is a solution of the generalized d'Alembert's long functional equation (2.3). In the following theorem under another kind of assumption we get that g is a solution of the generalized d'Alembert's short functional equation (2.2).

Theorem 3.1. *Let σ be a continuous involution of G . Let μ be a discrete complex measure with compact support such that μ is σ -invariant and $\mu * \mu = \mu$. Let $\delta \geq 0$. If the pair $f, g: G \rightarrow \mathbb{C}$, where f is an unbounded μ -biinvariant continuous solution of the following inequality*

$$(3.1) \quad \left| \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y) \right| \leq \delta$$

for all $x, y \in G$. Then g is a solution of the generalized d'Alembert's short functional equation (2.2)

Proof. Assume that f, g satisfy inequality (3.1) where f is unbounded on G . So, for all $x, y \in G$ we have

$$\begin{aligned} & \left| \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y) \right| \leq \delta, \\ & \left| \int_G f(xt\sigma(y))d\mu(t) + \int_G f(xty)d\mu(t) - 2f(x)g(\sigma(y)) \right| \leq \delta, \end{aligned}$$

and by triangle inequality we find

$$|2f(x)||g(y) - g(\sigma(y))| \leq 2\delta$$

for all $x, y \in G$. Since f is assumed to be unbounded, then we get $g(\sigma(y)) = g(y)$ for all $y \in G$. In the rest of the proof we use some ideas of the proof of Theorem 2.1 and Theorem 2.2.

First Case: We assume that the function $x \mapsto f(x) + f(\sigma(x))$ is a bounded function on G , that is $|f(x) + f(\sigma(x))| \leq \beta$ for some $\beta \geq 0$ and for all $x \in G$. Let $\Psi(x, y) = \int_G g(xty)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) - 2g(x)g(y)$, $x, y \in G$. We

will show that the function $(x, y, z) \mapsto 2f(z)\Psi(x, y) + 2f(y)\Psi(x, z)$ is a bounded function when g is bounded. The above computations show that

$$\begin{aligned} & 2f(z)\Psi(x, y) + 2f(y)\Psi(x, z) \\ &= 2g(x) \left[\int_G f(zt\sigma(y))d\mu(t) + \int_G f(yt\sigma(z))d\mu(t) \right] \\ & \quad - \int_G \int_G f(ytxs\sigma(z))d\mu(t)d\mu(s) - \int_G \int_G f(zs\sigma(x)t\sigma(y))d\mu(t)d\mu(s) \\ & \quad - \int_G \int_G f(zsxt\sigma(y))d\mu(t)d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))d\mu(t)d\mu(s). \end{aligned}$$

So, we get

$$(3.2) \quad |2f(z)\Psi(x, y) + 2f(y)\Psi(x, z)| \leq 2\beta\|\mu\|\|g(x)\| + 2\beta\|\mu\|^2$$

for all $x, y, z \in G$, where $\|\mu\| = \sup\{| \langle f, \mu \rangle |, f \in C(G) \mid \|f\|_\infty = 1\}$. There are two possibilities. One is: g is unbounded, then from [[8], Corollary 2.7 (iii)] and Theorem 2.2, we conclude that g is a solution of the generalized d'Alembert's short functional equation (2.1).

The other possibility is: g is a bounded function, then from (3.2) the function $(x, y, z) \mapsto 2f(z)\Psi(x, y) + 2f(y)\Psi(x, z)$ is a bounded function on G . Having assumed f unbounded, this implies that then there exists a sequence $(z_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} |f(z_n)| = +\infty$. By using (3.2), there exists $c_x \in \mathbb{C}$ such that $\Psi(x, y) = c_x f(y)$, and the function $(x, y, z) \mapsto 2f(z)c_x f(y) + 2f(y)c_x f(z)$ is bounded. By using again the unboundedness of f we get $c_x = 0$ for all $x \in G$. This means that $2f(z)\Psi(x, y) + 2f(y)\Psi(x, z) = 0$ for all $x, y, z \in G$. The computations above shows that g is a solution of the generalized d'Alembert's short functional equation (2.2).

Now, let f, g be functions such that f is unbounded on G and the function $(x, y) \mapsto \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)$ is bounded on $G \times G$. One can verify that the function: $(x, y) \mapsto \int_G f_\mu(xty)d\mu(t) + \int_G f_\mu(xt\sigma(y))d\mu(t) - 2f_\mu(x)g(y)$ is bounded on $G \times G$. If f_μ is unbounded and $f_\mu(e) = 0$, then since $\mu * \mu = \mu$ we have $f_\mu(x) + f_\mu(\sigma(x))$ is a bounded function, so by using the precedent proof, we get g satisfies equation (2.2). For the rest of the proof we fixe g and we assume that $f_\mu(e) \neq 0$. Replacing f_μ by $f_\mu/f_\mu(e)$ we may assume that $f_\mu(e) = 1$. Consider the function $\delta_a f(x) = \int_G f(atx)d\mu(t)$, $x \in G$. We can easily verify that the function $(x, y) \mapsto \int_G \delta_a f(xty)d\mu(t) + \int_G \delta_a f(xt\sigma(y))d\mu(t) - 2\delta_a f(x)g(y)$ is bounded on $G \times G$.

If there exists $a \in G$ such that $h = (\delta_a f)_\mu - (\delta_a f)_\mu(e)f_\mu$ is unbounded on G , since $h(e) = 0$, $h_\mu = h$, the function $(x, y) \mapsto \int_G h(xty)d\mu(t) + \int_G h(xt\sigma(y))d\mu(t) - 2h(x)g(y)$ is bounded we get $x \mapsto h(x) + h(\sigma(x))$ is a bounded function on G . Then from above computations we get that g is a solution of equation (2.2). Now, assume that for all $x \in G$ the function $y \mapsto (\delta_x f)_\mu(y) - (\delta_x f)_\mu(e)f_\mu(y)$ is bounded, that is there exists $M(x)$ such

that

$$(3.3) \quad \left| \int_G f(xty)d\mu(t) - \int_G f(xt)d\mu(t) \int_G f(ty)d\mu(t) \right| \leq M(x)$$

for all $x, y \in G$. Since f is assumed to be μ -biinvariant, that is $\int_G f(xt)d\mu(t) = \int_G f(tx)d\mu(t) = f(x)$ for all $x \in G$ then inequality (3.3) can be replaced by

$$(3.4) \quad \left| \int_G f(xty)d\mu(t) - f(x)f(y) \right| \leq M(x)$$

for all $x, y \in G$. Indeed in view of triangle inequality we get for all $x, y, z \in G$ that

$$\begin{aligned} |f(z)| \left| \int_G f(xty)d\mu(t) - f(x)f(y) \right| &\leq \left| - \int_G \int_G f(xtysz)d\mu(t)d\mu(s) + \int_G f(xty)d\mu(t)f(z) \right| \\ &+ \left| \int_G \int_G f(xtysz)d\mu(t)d\mu(s) - f(x) \int_G f(ysz)d\mu(s) \right| + |f(x)| \left| \int_G f(ysz)d\mu(s) - f(y)f(z) \right| \\ &\leq \int_G M(xty)d|\mu|(t) + \|\mu\|M(x) + |f(x)|M(y). \end{aligned}$$

Since f is assumed to be unbounded, then we get $\int_G f(xty)d\mu(t) = f(x)f(y)$ for all $x, y \in G$. Substituting this result into inequality (3.1) we get that

$$|f(x)| |f(y) + f(\sigma(y)) - 2g(y)| \leq \delta$$

for all $x, y \in G$. Since f is unbounded then we obtain $g(y) = \frac{f(y)+f(\sigma(y))}{2}$ for all $y \in G$, from which a simple computation shows that g is a solution of the generalized d'Alembert's short functional equation (2.2). This completes the proof. \square

The first part of the above proof proves the following corollary.

Corollary 3.2. *Let σ be a continuous involution of G . Let μ be a complex measure with compact support such that μ is σ -invariant and $\mu * \mu = \mu$. Let $\delta \geq 0$. If the pair $f, g: G \rightarrow \mathbb{C}$, where f is an unbounded continuous solution of the following inequality*

$$(3.5) \quad \left| \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y) \right| \leq \delta$$

for all $x, y \in G$ and such that $x \rightarrow f(x) + f(\sigma(x))$ is a bounded function. Then g is a solution of the generalized d'Alembert's short functional equation (2.2).

Corollary 3.3. *Let σ be a continuous involution of G . Let μ be a complex measure with compact support such that μ is σ -invariant and $\mu * \mu = \mu$. Let $\delta \geq 0$. If the pair $f, g: G \rightarrow \mathbb{C}$, where g is an unbounded continuous solution of the following inequality*

$$(3.6) \quad \left| \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y) \right| \leq \delta$$

for all $x, y \in G$. Then the pair f, g is a continuous solution of the functional equation (2.1). Furthermore, if $f \neq 0$ and $f_\mu \neq 0$, then g satisfies the generalized d'Alembert's short functional equation (2.2).

Proof. We use [[8], Corollary 2.7, (iii)] and Theorem 2.2. \square

In [7] on general groups and under the hypotheses that the function $(x, y) \rightarrow f(xy) + f(x\sigma(y)) - 2f(x)g(y)$ is a bounded function on $G \times G$, the function g satisfies d'Alembert's long functional equation

$$(3.7) \quad g(xy) + g(x\sigma(y)) + g(yx) + g(\sigma(y)x) = 4g(x)g(y), \quad x, y \in G$$

The following corollaries finishes the work on the Hyers Ulam stability of Wilson's functional equation

$$(3.8) \quad f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G$$

on groups when the solutions are unbounded functions.

If we let $\mu = \delta_e$: The dirac measure concentrated at the identity element of G , we apply Theorem 2.3 to $\mu = \delta_e$ to obtain the following result which has been proved by several authors in the case where G is an abelian group.

Corollary 3.4. *Let G be a group. Let σ be an involution of G . Let $\delta \geq 0$. If the pair $f, g: G \rightarrow \mathbb{C}$, where f is an unbounded solution of the following inequality*

$$(3.9) \quad |f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \delta$$

for all $x, y \in G$. Then g is a solution of d'Alembert's short functional equation

$$(3.10) \quad g(xy) + g(x\sigma(y)) = 2g(x)g(y), \quad x, y \in G$$

Now, we can ready to formulate the Hyers-Ulam stability of the classical Wilson's functional equation (3.8) on groups. The following result was obtained by Kannappan and Kim [19] under the condition that f is even and f satisfies the Kannappan condition $f(xyz) = f(yxz)$ for all $x, y, z \in G$.

Corollary 3.5. *Let $\delta \geq 0$, G a group, σ an involution of G . Suppose that the pair $f, g: G \rightarrow \mathbb{C}$ satisfies*

$$(3.11) \quad |f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \delta, \quad \text{for all } x, y \in G.$$

Under these assumptions the following statements hold:

- (1) *If f is unbounded, then g satisfies d'Alembert's short functional equation (3.10)*
- (2) *If g is unbounded and $f \neq 0$ then the pair (f, g) satisfies Wilson's functional equation (3.8) and g satisfies d'Alembert's short functional equation (3.10).*

Proof. We use [[7], Theorem 2.2] and Theorem 2.2. \square

Recently, the authors proved the following result.

Theorem 3.6. [7] *Let $\delta \geq 0$, G a group, χ a unitary character of G and σ an involution of G such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. Suppose that the pair $f, g : G \rightarrow \mathbb{C}$ satisfies*

$$(3.12) \quad |f(xy) + \chi(y)f(x\sigma(y)) - 2f(x)g(y)| \leq \delta, \text{ for all } x, y \in G.$$

Under these assumptions the following statements hold:

(a) *If f is unbounded then g satisfies d'Alembert's long functional equation*

$$(3.13) \quad g(xy) + \chi(y)g(x\sigma(y)) + g(yx) + \chi(y)g(\sigma(y)x) = 4g(x)g(y), \quad x, y \in G$$

(b) *If g is unbounded and $f \neq 0$, then the pair (f, g) satisfies Wilson's functional equation*

$$(3.14) \quad f(xy) + \chi(y)f(x\sigma(y)) = 2f(x)g(y) \quad x, y \in G$$

and g satisfies the d'Alembert's short functional equation

$$(3.15) \quad g(xy) + \chi(y)g(x\sigma(y)) = 2g(x)g(y) \quad x, y \in G.$$

The purpose in the following is to prove that in Theorem 3.6, case (a), the function g satisfies d'Alembert's short functional equation (3.15). We notice here that the solutions of equation (3.15) are obtained by Stetkær in [27].

Proposition 3.7. *Let G be a group. Let $\delta \geq 0$. Let χ a unitary character of G and σ an involution of G such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. If the pair $f, g : G \rightarrow \mathbb{C}$, where f is an unbounded solution of the following inequality*

$$(3.16) \quad |f(xy) + \chi(y)f(x\sigma(y)) - 2f(x)g(y)| \leq \delta$$

for all $x, y \in G$. Then g is a solution of d'Alembert's short functional equation (3.15).

Proof. In the proof we use similar reasoning to that in the proof of Theorem 2.3 and some author's computations [7]. Assume that the pair f, g satisfies inequality (3.16) where f is an unbounded function on G . First Case: We assume that the function $x \mapsto f(x) + \chi(x)f(\sigma(x))$ is a bounded function on G , that is $|f(x) + \chi(x)f(\sigma(x))| \leq \beta$ for some $\beta \geq 0$ and for all $x \in G$. Let $\Psi(x, y) = g(xy) + \chi(y)g(\sigma(y)x) - 2g(x)g(y)$, $x, y \in G$. We will show that the function $(x, y, z) \mapsto 2f(z)\Psi(x, y) + 2f(y)\Psi(x, z)$ is a bounded function in the special case when g is bounded. The computations in [7] show that

$$\begin{aligned} & 2f(z)\Psi(x, y) + 2f(y)\Psi(x, z) \\ &= \mu(y)2f(z\sigma(y))g(x) + \mu(z)2f(y\sigma(z))g(x) - \mu(z)f(yx\sigma(z)) \\ & \quad - \mu(x)\mu(y)f(z\sigma(x)\sigma(y)) - \mu(y)f(zx\sigma(y)) - \mu(x)\mu(z)f(y\sigma(x)\sigma(z)) \end{aligned}$$

So, we get

$$(3.17) \quad |2f(z)\Psi(x, y) + 2f(y)\Psi(x, z)| \leq 2\beta\|\mu\|\|g(x)\| + 2\beta\|\mu\|^2$$

for all $x, y, z \in G$. Now, we will discuss two subcases:

If g is unbounded then from [[7], Theorem 2.2, (b)], we conclude that g is a

solution of the generalized d'Alembert's short functional equation (3.15). In the rest of the proof we examine the case of g a bounded function on G . Then from (3.17) the function $(x, y, z) \mapsto 2f(z)\Psi(x, y) + 2f(y)\Psi(x, z)$ is a bounded function on G . By using similar computations in the proof of Theorem 2.3 we get our result and this completes the proof. \square

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